

INTERSECTION NUMBERS AND TWISTED PERIOD RELATIONS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION ${}_{m+1}F_m$

YOSHIAKI GOTO

ABSTRACT. We study the generalized hypergeometric function ${}_{m+1}F_m$ and the differential equation ${}_{m+1}E_m$ satisfied by it. We use the twisted (co)homology groups associated with an integral representation of Euler type. We evaluate the intersection numbers of some twisted cocycles which are defined as m -th exterior products of logarithmic 1-forms. We also give twisted cycles corresponding to the series solutions to ${}_{m+1}E_m$, and evaluate the intersection numbers of them. These intersection numbers of the twisted (co)cycles lead twisted period relations which give relations for two fundamental systems of solutions to ${}_{m+1}E_m$.

1. INTRODUCTION

The generalized hypergeometric function ${}_{m+1}F_m$ of a variable x with complex parameters $a_0, \dots, a_m, b_1, \dots, b_m$ is defined by

$${}_{m+1}F_m \left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_0, n) \cdots (a_m, n)}{(b_1, n) \cdots (b_m, n) n!} x^n,$$

where $b_1, \dots, b_m \notin \{0, -1, -2, \dots\}$ and $(c, n) = \Gamma(c+n)/\Gamma(c)$. This series converges in the unit disk $|x| < 1$, and satisfies the generalized hypergeometric differential equation

$${}_{m+1}E_m = {}_{m+1}E_m \left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix} \right) : \left[\theta \prod_{i=1}^m (\theta + b_i - 1) - x \prod_{j=0}^m (\theta - a_j) \right] f(x) = 0,$$

where $\theta = x \frac{d}{dx}$. The linear differential equation ${}_{m+1}E_m$ is of rank $m+1$ with regular singular points $x = 0, 1$, and ∞ . If $b_i - b_j \notin \mathbb{Z}$ ($0 \leq i < j \leq m$), a fundamental system of solutions to ${}_{m+1}E_m$ around $x = 0$ is given by the following $m+1$ functions:

$$(1) \quad \begin{aligned} f_0 &:= {}_{m+1}F_m \left(\begin{matrix} a_1, \dots, a_{m+1} \\ b_1, \dots, b_m \end{matrix} ; x \right), \\ f_r &:= x^{1-b_r} \cdot {}_{m+1}F_m \left(\begin{matrix} a_0 - b_r + 1, \dots, a_m - b_r + 1 \\ b_1 - b_r + 1, \dots, 2 - b_r, \dots, b_m - b_r + 1 \end{matrix} ; x \right), \end{aligned}$$

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where $1 \leq r \leq m$. It is known that ${}_{m+1}F_m$ admits the integral representation of Euler type:

$$\begin{aligned}
 (2) \quad & {}_{m+1}F_m \left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix} ; x \right) \\
 &= \prod_{i=1}^m \frac{\Gamma(b_i)}{\Gamma(a_i)\Gamma(b_i - a_i)} \int_D \prod_{j=1}^{m-1} \left(t_j^{a_j - b_{j+1}} (t_j - t_{j+1})^{b_{j+1} - a_{j+1} - 1} \right) \\
 &\quad \cdot t_m^{a_m - 1} (1 - t_1)^{b_1 - a_1 - 1} (1 - xt_m)^{-a_0} dt_1 \wedge \dots \wedge dt_m,
 \end{aligned}$$

where $D := \{(t_1, \dots, t_m) \in \mathbb{R}^m \mid 0 < t_m < t_{m-1} < \dots < t_1 < 1\}$. The branch of the integrand is defined by the principal value for x near to 0.

In this paper, we consider the twisted (co)homology groups associated with the integral representation (2). Note that the singular locus of the integrand of (2) is not normally crossing. In such a case, as is studied in [8], the resolution of singularities is an effective way for the study of intersections of the twisted (co)homology groups. However, our singularities are so complicated for a general m that it seems difficult to resolute them. To conquer this difficulty, we find a systematic method completing the resolution of the singular locus. We blow up the singular locus step by step and use the combinatorial structure of divisors which should exist in the complete resolution. The resolution of the singular locus enables us to evaluate intersection numbers for twisted cocycles. We give formulas for the intersection numbers of m -th exterior products of logarithmic 1-forms, which span the twisted cohomology group. For the study of the twisted homology group, we avoid the complexity of the resolution. We construct twisted cycles corresponding to the $m+1$ solutions (1) to ${}_{m+1}E_m$ by using the method given in [5] and [6]. They are made by the bounded chambers, and their boundaries are canceled by different ways from the usual regularization. It is an advantage of our construction that we can evaluate their intersection numbers by the formula in [12] for a normally crossing singular locus. Intersection numbers of twisted homology and cohomology groups imply twisted period relations for two fundamental systems of solutions to ${}_{m+1}E_m$ with different parameters. These relations are transformed into quadratic relations among hypergeometric series ${}_{m+1}F_m$'s. Since our intersection matrices are diagonal, it is easy to reduce the twisted period relations to quadratic relations among ${}_{m+1}F_m$'s.

In [10], twisted cycles corresponding to the solutions (1) to ${}_{m+1}E_m$ are obtained from real (non-bounded) chambers, and their intersection numbers are evaluated by the method in [8]. Since these cycles are scalar multiples of ours as elements of the twisted homology group, we give their explicit correspondence in Remark 4.8. Twisted period relations for ${}_{m+1}F_m$ are given in [11] by the study of the intersection forms of (co)homology groups with coefficients in the local system of rank m given as the solution space to ${}_{m+1}E_m$. Another integral representation of ${}_{m+1}F_m$ and its inductive structure are used in [11].

As is in [2], the irreducibility condition of the differential equation ${}_{m+1}E_m$ is known to be $a_i - b_j \notin \mathbb{Z}$ ($0 \leq i, j \leq m$), where we put $b_0 := 0$ (though b_0 is usually defined by 1, we use this setting for our convenience). Since we use the fundamental system (1) of solutions to ${}_{m+1}E_m$, we assume throughout this paper that the parameters a_i, b_j satisfy the condition

$$(3) \quad a_i - b_j \notin \mathbb{Z} \ (0 \leq i, j \leq m), \quad b_i - b_j \notin \mathbb{Z} \ (0 \leq i < j \leq m).$$

2. TWISTED (CO)HOMOLOGY GROUPS ASSOCIATED WITH THE INTEGRAL REPRESENTATION (2)

For twisted homology groups, twisted cohomology groups, and the intersection forms, refer to [1], [12], or [5]. We use the same notations as in [5] and [6].

In this paper, we mainly consider the twisted (co)homology group in [5] for

$$M := \mathbb{C}^m - \left(\bigcup_{j=1}^m (t_j = 0) \cup \bigcup_{j=2}^m (t_{j-1} - t_j = 0) \cup (1 - t_1 = 0) \cup (1 - xt_m = 0) \right)$$

and the multi-valued function

$$u := \prod_{j=1}^m t_j^{a_j - b_{j+1}} \cdot \prod_{j=2}^m (t_{j-1} - t_j)^{b_j - a_j} \cdot (1 - t_1)^{b_1 - a_1} \cdot (1 - xt_m)^{-a_0}.$$

We put $\omega := d \log u$, where d is the exterior derivative with respect to the variables t_1, \dots, t_m (not to x regarded as a parameter). The twisted cohomology group, that with compact support, and the twisted homology group are denoted by $H^k(M, \nabla_\omega)$, $H_c^k(M, \nabla_\omega)$, and $H_k(\mathcal{C}_\bullet(M, u))$, respectively. Here, ∇_ω is the covariant differential operator defined as $\nabla_\omega := d + \omega \wedge$. The expression (2) means that the integral

$$\int_{D \otimes u} u \varphi_0, \quad \varphi_0 := \frac{dt_1 \wedge \dots \wedge dt_m}{t_m(1 - t_1)(t_1 - t_2) \dots (t_{m-1} - t_m)}$$

represents ${}_{m+1}F_m$ modulo Gamma factors. By [1] and [3], we have $H^k(M, \nabla_\omega) = 0$ ($k \neq m$), $\dim H^m(M, \nabla_\omega) = m + 1$, and there is a canonical isomorphism

$$j : H^m(M, \nabla_\omega) \rightarrow H_c^m(M, \nabla_\omega).$$

By the Poincaré duality, we have

$$\begin{aligned} \dim H_k(\mathcal{C}_\bullet(M, u)) &= \dim H^k(M, \nabla_\omega) = 0 \quad (k \neq m), \\ \dim H_m(\mathcal{C}_\bullet(M, u)) &= \dim H^m(M, \nabla_\omega) = m + 1. \end{aligned}$$

The intersection form I_h on the twisted homology groups is the pairing between $H_m(\mathcal{C}_\bullet(M, u))$ and $H_m(\mathcal{C}_\bullet(M, u^{-1}))$. The intersection form I_c on the twisted cohomology groups is the pairing between $H_c^m(M, \nabla_\omega)$ and $H^m(M, \nabla_{-\omega})$. By using j , we can regard the intersection form I_c as the pairing between $H^m(M, \nabla_\omega)$ and $H^m(M, \nabla_{-\omega})$, i.e.,

$$I_c(\psi, \psi') := \int_M j(\psi) \wedge \psi', \quad \psi \in H^m(M, \nabla_\omega), \quad \psi' \in H^m(M, \nabla_{-\omega}).$$

3. TWISTED COHOMOLOGY GROUPS AND INTERSECTION NUMBERS

In this section, we give two systems of twisted cocycles, and evaluate their intersection numbers.

We embed M into the projective space \mathbb{P}^m , that is, we regard M as the open subset of \mathbb{P}^m :

$$M = \mathbb{P}^m - \left(\bigcup_{j=0}^m L_j \cup \bigcup_{j=0}^m H_j \right) \subset \mathbb{C}^m \subset \mathbb{P}^m,$$

where

$$\begin{aligned} L_j &:= (T_j = 0) \quad (0 \leq j \leq m), \\ H_j &:= (T_{j-1} - T_j = 0) \quad (1 \leq j \leq m), \quad H_0 := (T_0 - xT_m = 0). \end{aligned}$$

By the homogeneous coordinates T_0, \dots, T_m , the multi-valued function u is expressed as

$$u = T_0^{\lambda_0} (T_0 - xT_m)^{\mu_0} \cdot \prod_{j=1}^m T_j^{\lambda_j} (T_{j-1} - T_j)^{\mu_j},$$

where

$$\begin{aligned} \lambda_j &:= a_j - b_{j+1} \quad (1 \leq j \leq m-1), \quad \lambda_m := a_m, \\ \mu_j &:= b_j - a_j \quad (1 \leq j \leq m), \quad \mu_0 := -a_0, \\ \lambda_0 &:= -\left(\sum_{j=1}^m \lambda_j + \sum_{j=0}^m \mu_j \right) = a_0 - b_1. \end{aligned}$$

Note that $L_0 = (T_0 = 0)$ is the hyperplane at infinity, i.e., $M \subset \mathbb{C}^m = \mathbb{P}^m - L_0 \subset \mathbb{P}^m$ and the coordinates t_1, \dots, t_m on \mathbb{C}^m are defined as $t_j = T_j/T_0$. Hereafter, we regard subscripts as elements in $\mathbb{Z}/(m+1)\mathbb{Z}$. For example, we have $a_{m+1} = a_0$, $b_{m+1} = b_0 = 0$, and

$$\lambda_j = a_j - b_{j+1}, \quad \mu_j = b_j - a_j \quad (0 \leq j \leq m).$$

Let ℓ_k and h_k ($0 \leq k \leq m$) be the defining linear forms of L_k and H_k , respectively. We define an m -form on M by

$$\phi(f_0, \dots, f_m) := d \log \left(\frac{f_0}{f_1} \right) \wedge d \log \left(\frac{f_1}{f_2} \right) \wedge \dots \wedge d \log \left(\frac{f_{m-1}}{f_m} \right)$$

for $f_0, \dots, f_m \in \{\ell_0, \dots, \ell_m, h_0, \dots, h_m\}$. We consider two systems $\{\varphi_k\}_{k=0}^m$ and $\{\psi_k\}_{k=0}^m$ given as

$$\begin{aligned} \varphi_k &:= \phi(h_0, \dots, h_{k-1}, \ell_{k-1}, h_{k+1}, \dots, h_m), \\ \psi_k &:= \phi(h_0, \dots, h_{k-1}, \ell_k, h_{k+1}, \dots, h_m). \end{aligned}$$

Using the coordinates $t_j = T_j/T_0$ ($1 \leq j \leq m$) of $\mathbb{C}^m = \mathbb{P}^m - L_0$, we have

$$\begin{aligned} \varphi_0 &= \frac{dt_1 \wedge \dots \wedge dt_m}{t_m(1-t_1)(t_1-t_2) \cdots (t_{m-1}-t_m)}, \\ \psi_0 &= \frac{dt_1 \wedge \dots \wedge dt_m}{(1-t_1)(t_1-t_2) \cdots (t_{m-1}-t_m)}, \\ \varphi_r &= \frac{xd t_1 \wedge \dots \wedge dt_m}{t_{r-1}(1-x t_m)(1-t_1)(t_1-t_2) \cdots \widehat{(t_{r-1}-t_r)} \cdots (t_{m-1}-t_m)}, \\ \psi_r &= \frac{dt_1 \wedge \dots \wedge dt_m}{t_r(1-x t_m)(1-t_1)(t_1-t_2) \cdots \widehat{(t_{r-1}-t_r)} \cdots (t_{m-1}-t_m)}, \end{aligned}$$

where $1 \leq r \leq m$. Note that the m -form φ_0 coincides with that defined in Section 2.

Theorem 3.1.

$$\begin{aligned}
(4) \quad & I_c(\varphi_i, \varphi_j) = I_c(\psi_i, \psi_j) = 0 \quad (i \neq j), \\
(5) \quad & I_c(\varphi_k, \varphi_k) = (2\pi\sqrt{-1})^m \prod_{\substack{0 \leq l \leq m \\ l \neq k}} \frac{b_l - b_k}{(a_l - b_k)(b_l - a_l)}, \\
(6) \quad & I_c(\psi_k, \psi_k) = (2\pi\sqrt{-1})^m \prod_{\substack{0 \leq l \leq m \\ l \neq k}} \frac{a_l - a_k}{(b_l - a_k)(b_l - a_l)}, \\
(7) \quad & I_c(\varphi_i, \psi_j) = I_c(\psi_j, \varphi_i) = \varepsilon_{ij} (2\pi\sqrt{-1})^m \frac{(b_i - a_i)(b_j - a_j)}{(b_i - a_j)} \prod_{l=0}^m \frac{1}{b_l - a_l},
\end{aligned}$$

where

$$\varepsilon_{ij} := \begin{cases} -1 & (i \neq j \text{ and } (i = 0 \text{ or } j = 0)), \\ 1 & (\text{otherwise}). \end{cases}$$

The following corollary follows from this theorem immediately.

Corollary 3.2. *Under the condition (3), $\varphi_0, \dots, \varphi_m$ form a basis of $H^m(M, \nabla_\omega)$.*

Proof. Let $C := (I_c(\varphi_i, \varphi_j))_{i,j=0,\dots,m}$ be the intersection matrix. Then we have

$$\det(C) = (2\pi\sqrt{-1})^{m(m+1)} \prod_{l=0}^m \frac{1}{(b_l - a_l)^m} \prod_{0 \leq i \neq j \leq m} \frac{b_i - b_j}{a_i - b_j},$$

which does not vanish under the condition (3). \square

In the remainder of this section, we prove Theorem 3.1. According to [9], to evaluate intersection numbers, we have to blow up \mathbb{P}^m so that the pole divisor of the pull back of $\omega = d \log u$ is normally crossing. And we need informations of the m -forms around the points at which m components of the pole divisor intersect.

For $i \neq j$, $j+1$, let $L_{j,j+1,\dots,i-1}$ be the exceptional divisor obtained by blowing up along $L_j \cap L_{j+1} \cap \dots \cap L_{i-1} = (T_j = T_{j+1} = \dots = T_{i-1} = 0)$. The residue of the pull-back of ω along $L_{j,j+1,\dots,i-1}$ is

$$\lambda_{j,j+1,\dots,i-1} = \sum_{l=j}^{i-1} \lambda_l + \sum_{l=j+1}^{i-1} \mu_l = \sum_{l=j}^{i-1} (a_l - b_{l+1}) + \sum_{l=j+1}^{i-1} (b_l - a_l) = a_j - b_i$$

(recall that the indices are regarded as elements in $\mathbb{Z}/(m+1)\mathbb{Z}$). Note that for example, L_{12} is an exceptional divisor, however L_1 is not.

First, we investigate the intersections of $L_1, \dots, L_m, H_0, \dots, H_m$, and exceptional divisors obtained by blowing up along $L_j \cap L_{j+1} \cap \dots \cap L_m$ ($1 \leq j \leq m-1$), in $\mathbb{C}^m = \mathbb{P}^m - L_0$. By a straightforward calculation in $\mathbb{P}^m - L_0$, we obtain the following lemma.

Lemma 3.3. *We blow up $\mathbb{P}^m - L_0$ along*

$$L_j \cap L_{j+1} \cap \dots \cap L_m \quad (1 \leq j \leq m-1).$$

In $\{\varphi_0, \dots, \varphi_m, \psi_0, \dots, \psi_m\}$, only φ_0 and ψ_j have $L_{j,j+1,\dots,m}$ as a component of the pole divisor. Further, we have

$$H_k \cap L_{j,j+1,\dots,m} = \emptyset \iff k = 0 \text{ or } k = j.$$

Second, we describe the all intersections of $L_0, \dots, L_m, H_0, \dots, H_m$, and exceptional divisors. We use the combinatorial structure of them, which arise from the similarity between the expression of u on $\mathbb{P}^m - L_0$ and that on $\mathbb{P}^m - L_k$.

Lemma 3.4. *After blowing up along all $L_j \cap L_{j+1} \cap \cdots \cap L_{i-1}$ (with a suitable order), the pole divisor of the pull-back of ω is normally crossing. Let Φ_k (resp. Ψ_k) be a set consisting of the components of the pole divisor of the pull-back of φ_k (resp. ψ_k). Then we have*

$$\begin{aligned}\Phi_k &= \{H_{k+1}, H_{k+2}, \dots, H_{k-1}, L_{k+1, k+2, \dots, k-1}, L_{k+2, \dots, k-1}, \dots, L_{k-2, k-1}, L_{k-1}\}, \\ \Psi_k &= \{H_{k+1}, H_{k+2}, \dots, H_{k-1}, L_k, L_{k, k+1}, \dots, L_{k, k+1, \dots, k-3}, L_{k, k+1, \dots, k-3, k-2}\}.\end{aligned}$$

Moreover, we have

$$H_k \cap L_{j, j+1, \dots, i-1} = \emptyset \iff k = i \text{ or } k = j.$$

Proof. Recall that u is expressed as

$$u = t_1^{\lambda_1} \cdots t_m^{\lambda_m} \cdot (1 - xt_m)^{\mu_0} (1 - t_1)^{\mu_1} \cdot (t_1 - t_2)^{\mu_2} \cdots (t_{m-1} - t_m)^{\mu_m}$$

on $\mathbb{C}^m = \mathbb{P}^m - L_0$ (with coordinates $t_j = T_j/T_0$ ($1 \leq j \leq m$)). On the other hand, on $\mathbb{P}^m - L_k$, it is expressed as

$$\begin{aligned}u &= s_0^{\lambda_0} \cdots s_{k-1}^{\lambda_{k-1}} s_{k+1}^{\lambda_{k+1}} \cdots s_m^{\lambda_m} \\ &\quad \cdot (s_0 - xs_m)^{\mu_0} (s_0 - s_1)^{\mu_1} \cdots (s_{k-1} - 1)^{\mu_k} (1 - s_{k+1})^{\mu_{k+1}} \cdots (s_{m-1} - s_m)^{\mu_m} \\ &= s_{k+1}^{\lambda_{k+1}} \cdots s_m^{\lambda_m} s_0^{\lambda_0} \cdots s_{k-1}^{\lambda_{k-1}} \cdot (s_{k-1} - 1)^{\mu_k} (1 - s_{k+1})^{\mu_{k+1}} \\ &\quad \cdot (s_{k+1} - s_{k+2})^{\mu_{k+2}} \cdots (s_{m-1} - s_m)^{\mu_m} (s_0 - xs_m)^{\mu_0} (s_0 - s_1)^{\mu_1} \cdots (s_{k-2} - s_{k-1})^{\mu_{k-1}},\end{aligned}$$

in terms of coordinates $s_j = T_j/T_k$ ($0 \leq j \leq m$, $j \neq k$). Thus,

$$L_{k+1}, \dots, L_m, L_0, \dots, L_{k-1} \text{ and } H_k, \dots, H_m, H_0, \dots, H_{k-1}$$

in $\mathbb{P}^m - L_k$ behave similarly to

$$L_1, \dots, L_m \text{ and } H_0, \dots, H_m$$

in $\mathbb{P}^m - L_0$. Then we obtain this lemma by Lemma 3.3. \square

Remark 3.5. *The slight differences come from the signs of $s_0 - xs_m$ and $s_{k-1} - 1$ at the intersection points. As mentioned below, these differences make complexity of ε_{ij} .*

In particular, we have $\#\Phi_k = \#\Psi_k = 2m$. We put

$$\Phi_k^{(m)} := \{\{D_1, \dots, D_m\} \subset \Phi_k \mid D_i \neq D_j \text{ } (i \neq j), D_1 \cap \cdots \cap D_m \neq \emptyset\}$$

($\Psi_k^{(m)}$ is also defined in a similar way). Then Lemma 3.4 implies

$$\Phi_k^{(m)} = \{\{H_p\}_{p \in I} \cup \{L_{q, q+1, \dots, k-1}\}_{q \notin I} \mid I \subset \{k+1, k+2, \dots, m, 0, \dots, k-1\}\},$$

$$\Psi_k^{(m)} = \{\{H_p\}_{p \in I} \cup \{L_{k, k+1, \dots, q-1}\}_{q \notin I} \mid I \subset \{k+1, k+2, \dots, m, 0, \dots, k-1\}\}.$$

Finally, we evaluate the intersection numbers of φ_i 's and ψ_j 's by using results in [9].

Proof of Theorem 3.1. First, we obtain (4), since it is clear that

$$\Phi_i^{(m)} \cap \Phi_j^{(m)} = \Psi_i^{(m)} \cap \Psi_j^{(m)} = \emptyset \quad (i \neq j).$$

Second, we have

$$\begin{aligned}I_c(\varphi_k, \varphi_k) &= (2\pi\sqrt{-1})^m \sum_{I \subset \{k+1, k+2, \dots, k-1\}} \prod_{i \in I} \frac{1}{\mu_i} \cdot \prod_{j \notin I} \frac{1}{\lambda_{j, j+1, \dots, k-1}} \\ &= (2\pi\sqrt{-1})^m \sum_{I \subset \{k+1, k+2, \dots, k-1\}} \prod_{i \in I} \frac{1}{b_i - a_i} \cdot \prod_{j \notin I} \frac{1}{a_j - b_k}.\end{aligned}$$

By induction on m , we can show that

$$\sum_{I \subset \{k+1, k+2, \dots, k-1\}} \prod_{i \in I} \frac{1}{b_i - a_i} \prod_{j \notin I} \frac{1}{a_j - b_k} = \prod_{\substack{0 \leq l \leq m \\ l \neq k}} \frac{b_l - b_k}{(a_l - b_k)(b_l - a_l)},$$

which implies (5). The equality (6) can be shown in a similar way. Finally, we prove (7). Because of

$$\begin{aligned} \Phi_i^{(m)} \cap \Psi_i^{(m)} &= \{\{H_0, \dots, H_m\} - \{H_i\}\}, \\ \Phi_i^{(m)} \cap \Psi_j^{(m)} &= \{\{H_0, \dots, H_m, L_{j,j+1}, \dots, i-1\} - \{H_i, H_j\}\} \quad (i \neq j), \end{aligned}$$

we have

$$\begin{aligned} I_c(\varphi_i, \psi_i) &= \varepsilon'_{ii} \cdot (2\pi\sqrt{-1})^m \prod_{l \neq i} \frac{1}{\mu_l} = \varepsilon'_{ii} \cdot (2\pi\sqrt{-1})^m \prod_{l \neq i} \frac{1}{b_l - a_l}, \\ I_c(\varphi_i, \psi_j) &= \varepsilon'_{ij} \cdot (2\pi\sqrt{-1})^m \cdot \frac{1}{\lambda_{j,j+1}, \dots, i-1}} \prod_{l \neq i, j} \frac{1}{\mu_l} \\ &= \varepsilon'_{ij} \cdot (2\pi\sqrt{-1})^m \cdot \frac{1}{a_j - b_i} \prod_{l \neq i, j} \frac{1}{b_l - a_l} \quad (i \neq j), \end{aligned}$$

where $\varepsilon'_{ij} = \pm 1$. Let us show that

$$(8) \quad \varepsilon'_{ij} = \begin{cases} 1 & (i = 0 \text{ or } j = 0 \text{ or } i = j), \\ -1 & (\text{otherwise}). \end{cases}$$

When we evaluate the intersection number $I_c(\varphi_i, \psi_j)$, it is sufficient to consider blowing up along only

$$L_{i+1} \cap L_{i+2} \cap \dots \cap L_{i-1}, \quad L_{i+2} \cap \dots \cap L_{i-1}, \dots, \quad L_{i-2} \cap L_{i-1}$$

in the coordinate system of $\mathbb{P}^m - L_i$, since the pole divisor of φ_i is normally crossing after this blowing-up process. Put $\Phi_i^{(m)} \cap \Psi_j^{(m)} = \{\{G_1, \dots, G_m\}\}$, and let g_l be the defining linear forms of G_l . By taking appropriate coordinates t'_1, \dots, t'_m , we express $\varphi_i \cdot \prod_l g_l$ and $\psi_j \cdot \prod_l g_l$ around the intersection point $G_1 \cap \dots \cap G_m$ explicitly.

(i) For $i = j = 0$, we have

$$\begin{aligned} g_l &= 1 - t'_l \quad (1 \leq l \leq m), \\ \varphi_0 \cdot \prod_l g_l &= \frac{dt'_1 \wedge \dots \wedge dt'_m}{t'_1 \dots t'_m}, \quad \psi_0 \cdot \prod_l g_l = dt'_1 \wedge \dots \wedge dt'_m, \end{aligned}$$

and the intersection point $G_1 \cap \dots \cap G_m$ is expressed as

$$t'_l = 1 \quad (1 \leq l \leq m).$$

(ii) For $i = 0$ and $j \neq 0$, we have

$$\begin{aligned} g_j &= t'_j, \quad g_l = 1 - t'_l \quad (l \neq j), \\ \varphi_0 \cdot \prod_l g_l &= \frac{dt'_1 \wedge \dots \wedge dt'_m}{t'_1 \dots \widehat{t'_j} \dots t'_m (1 - t'_j)}, \quad \psi_j \cdot \prod_l g_l = \frac{dt'_1 \wedge \dots \wedge dt'_m}{1 - xt'_1 \dots t'_m}, \end{aligned}$$

and the intersection point $G_1 \cap \dots \cap G_m$ is expressed as

$$t'_j = 0, \quad t'_l = 1 \quad (l \neq j).$$

(iii) For $i \neq 0$ and $j = i$, we have

$$\begin{aligned} g_{m+1-i} &= t'_{m+1-i} - x, \quad g_l = 1 - t'_l \quad (l \neq m+1-i), \\ \varphi_i \cdot \prod_l g_l &= x \cdot \frac{dt'_1 \wedge \dots \wedge dt'_m}{t'_1 \dots t'_m}, \quad \psi_0 \cdot \prod_l g_l = dt'_1 \wedge \dots \wedge dt'_m, \end{aligned}$$

and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_{m+1-i} = x, \quad t'_l = 1 \quad (l \neq m+1-i).$$

(iv) For $i \neq 0$ and $j = 0$, we have

$$\begin{aligned} g_{m+1-i} &= t'_{m+1-i}, \quad g_l = 1 - t'_l \quad (l \neq m-i+1), \\ \varphi_i \cdot \prod_l g_l &= x \cdot \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots \widehat{t'_{m+1-i}} \cdots t'_m (t'_{m+1-i} - x)}, \\ \psi_0 \cdot \prod_l g_l &= \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots t'_m - 1} \end{aligned}$$

and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_{m+1-i} = 0, \quad t'_l = 1 \quad (l \neq m+1-i).$$

(v) For $i \neq 0$ and $j \neq 0, i$, we have

$$\begin{aligned} g_{j-i} &= t'_{j-i}, \quad g_{m+1-i} = t'_{m+1-i} - x, \quad g_l = 1 - t'_l \quad (l \neq j-i, m+1-i), \\ \varphi_i \cdot \prod_l g_l &= x \cdot \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots \widehat{t'_{j-i}} \cdots t'_m (1 - t'_{j-i})}, \quad \psi_0 \cdot \prod_l g_l = \frac{dt'_1 \wedge \cdots \wedge dt'_m}{t'_1 \cdots t'_m - 1} \end{aligned}$$

(note that if $j < i$ then we regard $j-i$ as $m+1+j-i$), and the intersection point $G_1 \cap \cdots \cap G_m$ is expressed as

$$t'_{j-i} = 0, \quad t'_{m+1-i} = x, \quad t'_l = 1 \quad (l \neq j-i, m+1-i).$$

Hence we have (8), and complete the proof of (7). \square

4. TWISTED HOMOLOGY GROUPS AND INTERSECTION NUMBERS

In this section, we construct $m+1$ twisted cycles in M corresponding to the solutions (1) to ${}_{m+1}E_m$.

For $0 \leq k \leq m$, we set

$$M_k := \mathbb{C}^m - \left(\bigcup_{j=1}^m \left((z_j = 0) \cup (1 - z_j = 0) \right) \cup \left(z_k - x \prod_{j \neq k} z_j = 0 \right) \right),$$

where z_1, \dots, z_m are coordinates of \mathbb{C}^m . Let u_k and ϕ_k be a multi-valued function and an m -form on M_k defined as

$$\begin{aligned} u_k &:= \prod_{j \neq k} z_j^{a_j - b_k} (1 - z_j)^{b_j - a_j} \cdot z_k^{a_k} (1 - z_k)^{-a_0} \left(z_k - x \prod_{j \neq k} z_j \right)^{b_k - a_k}, \\ \phi_k &:= \frac{dz_1 \wedge \cdots \wedge dz_m}{z_k \cdot \prod_{j \neq k} (1 - z_j) \cdot (z_k - x \prod_{j \neq k} z_j)}, \end{aligned}$$

respectively. Here, we regard z_0 as 1; we have

$$\begin{aligned} z_0 - x \prod_{j \neq k} z_j &= 1 - x \prod_{i=1}^m z_i, \\ u_0 &= \prod_{i=1}^m z_i^{a_i} (1 - z_i)^{b_i - a_i} \cdot \left(1 - x \prod_{i=1}^m z_i \right)^{-a_0}, \quad \phi_0 = \frac{dz_1 \wedge \cdots \wedge dz_m}{\prod_{i=1}^m (z_i (1 - z_i))}. \end{aligned}$$

We construct a twisted cycle $\tilde{\Delta}_k$ loaded by u_k in M_k . Let x and ε be positive real numbers satisfying

$$\varepsilon < \frac{1}{2}, \quad x < \frac{\varepsilon}{(1 + \varepsilon)^{m-1}}$$

(for example, if

$$\varepsilon = \frac{1}{3}, \quad 0 < x < \frac{1}{3} \cdot \left(\frac{3}{4}\right)^{m-1},$$

this condition holds). Thus the direct product

$$\sigma_k := \{(z_1, \dots, z_m) \in \mathbb{R}^m \mid \varepsilon \leq z_k \leq 1 - \varepsilon \ (1 \leq r \leq m)\}$$

of m intervals is contained in the bounded domain

$$\left\{ (z_1, \dots, z_m) \in \mathbb{R}^m \mid 0 < z_j < 1, \ z_k > x \cdot \prod_{j \neq k} z_j \right\} \subset (0, 1)^m.$$

The orientation of σ_k is induced from the natural embedding $\mathbb{R}^m \subset \mathbb{C}^m$.

By using the ε -neighborhoods of $C_1 := (z_1 = 0)$, \dots , $C_m := (z_m = 0)$, $C_{m+1} := (1 - z_1 = 0)$, \dots , $C_{2m} := (1 - z_m = 0)$, we construct a twisted cycle $\tilde{\Delta}_k$ from $\sigma_k \otimes u_k$ in a similar way in [5] and [6]. If $k \neq 0$, we have to consider the difference of branches of

$$z_k^{a_k} \left(z_k - x \prod_{j \neq k} z_j \right)^{b_k - a_k}$$

at the ending and starting points of a circle surrounding C_k . Indeed, for fixed positive real numbers z_j ($j \neq k$), the solution $x \prod_{j \neq k} z_j$ of the equation $z_k - x \prod_{j \neq k} z_j = 0$ belongs to \mathbb{R} and satisfies

$$x \cdot \prod_{j \neq k} z_j < x < \varepsilon.$$

Thus, the difference is $\exp(2\pi\sqrt{-1}a_k) \exp(2\pi\sqrt{-1}(b_k - a_k))$ and the exponent about this contribution is

$$a_k + (b_k - a_k) = b_k.$$

The exponents about the contributions of the circles surrounding C_{m+k} , C_j , C_{m+j} ($j \neq k$) are simply

$$-a_0, \ a_j - b_k, \ b_j - a_j,$$

respectively.

Remark 4.1. If $k = 0$, the exponents about the contributions of the circles surrounding C_i , C_{m+i} ($1 \leq i \leq m$) are simply $a_i - b_0 = a_i$, $b_i - a_i$, respectively. Since $\overline{(0, 1)^m} \cap \{z \mid 1 - x \prod_i z_i = 0\} = \emptyset$, the twisted cycle $\tilde{\Delta}_0$ is the usual regularization of $(0, 1)^m \otimes u_0$.

Proposition 4.2.

$$\begin{aligned} \int_{\tilde{\Delta}_0} u_0 \phi_0 &= \prod_{i=1}^m \frac{\Gamma(a_i) \Gamma(b_i - a_i)}{\Gamma(b_i)} \cdot {}_{m+1}F_m \left(\begin{matrix} a_0, \dots, a_m \\ b_1, \dots, b_m \end{matrix} ; x \right), \\ \int_{\tilde{\Delta}_r} u_r \phi_r &= \frac{\Gamma(b_r - 1) \Gamma(1 - a_0)}{\Gamma(b_r - a_0)} \cdot \prod_{\substack{1 \leq j \leq m \\ j \neq r}} \frac{\Gamma(a_j - b_r + 1) \Gamma(b_j - a_j)}{\Gamma(b_j - b_r + 1)} \\ &\quad \cdot {}_{m+1}F_m \left(\begin{matrix} a_0 - b_r + 1, \dots, a_m - b_r + 1 \\ b_1 - b_r + 1, \dots, 2 - b_r, \dots, b_m - b_r + 1 \end{matrix} ; x \right) \quad (1 \leq r \leq m). \end{aligned}$$

Proof. In a similar way to Proposition 4.3 of [5] or Proposition 4.3 of [6], we can show this proposition by expanding the left hand sides with respect to x . Note that we use the equalities

$$\begin{aligned} \int_{\tilde{\Delta}_0} \prod_{i=1}^m z_i^{a_i+n-1} (1-z_i)^{b_i-a_i-1} dz &= \prod_{i=1}^m \frac{\Gamma(a_i+n)\Gamma(b_i-a_i)}{\Gamma(b_i+n)}, \\ \int_{\tilde{\Delta}_r} \prod_{j \neq r} z_j^{a_j-b_r+n} (1-z_j)^{b_j-a_j-1} \cdot z_r^{b_r-2-n} (1-z_r)^{-a_0} dz \\ &= \prod_{j \neq r} \frac{\Gamma(a_j-b_r+n+1)\Gamma(b_j-a_j)}{\Gamma(b_j-b_r+n+1)} \cdot \frac{\Gamma(b_r-1-n)\Gamma(1-a_0)}{\Gamma(b_r-a_0-n)}, \end{aligned}$$

for a natural number n and $1 \leq r \leq m$. The second equality follows from the fact that the twisted cycle $\tilde{\Delta}_r$ of the integral can be identified with the usual regularization of the domain $(0,1)^m$ loaded by the multi-valued function

$$\prod_{j \neq r} z_j^{a_j-b_r+n} (1-z_j)^{b_j-a_j-1} \cdot z_r^{b_r-2-n} (1-z_r)^{-a_0}$$

on $\mathbb{C}^m - \bigcup_{j=1}^m ((z_j = 0) \cup (1 - z_j = 0))$. □

We define a bijection $\iota_k : M_k \rightarrow M$ by

$$\begin{aligned} \iota_0(z_1, \dots, z_m) &:= (t_1, \dots, t_m); \quad t_s = \prod_{i=1}^s z_i, \\ \iota_r(z_1, \dots, z_m) &:= (t_1, \dots, t_m); \quad t_s = \prod_{j=1}^s z_j \quad (s < r), \quad t_s = \frac{z_r}{x \cdot \prod_{j=s+1}^m z_j} \quad (s \geq r), \end{aligned}$$

where $1 \leq r \leq m$. We also define branches of the multi-valued function u on real chambers in M . Let $D_r \subset \mathbb{R}^m$ ($1 \leq r \leq m$) be the chamber defined by

$$t_j > 0 \quad (1 \leq j \leq m), \quad 1 - xt_m > 0, \quad t_{j-1} - t_j > 0 \quad (j \neq r), \quad t_{r-1} - t_r < 0,$$

where we regard t_0 as 1. On D_r , the arguments of the factors of u are given as follows.

t_j	$1 - xt_m$	$t_{j-1} - t_j \quad (j \neq r)$	$t_{r-1} - t_r$
0	0	0	$-\pi$

Recall that on $D = \{(t_1, \dots, t_m) \in \mathbb{R}^m \mid 0 < t_m < t_{m-1} < \dots < t_1 < 1\}$, all of the arguments of the factors of u are 0.

Theorem 4.3. *We define a twisted cycle Δ_k in M by*

$$\Delta_k := (\iota_k)_*(\tilde{\Delta}_k).$$

Then we have

$$\begin{aligned} \int_{\Delta_0} u \varphi_0 &= \prod_{i=1}^m \frac{\Gamma(a_i)\Gamma(b_i-a_i)}{\Gamma(b_i)} \cdot f_0, \\ \int_{\Delta_r} u \varphi_0 &= e^{-\pi\sqrt{-1}(b_r-a_r-1)} \frac{\Gamma(b_r-1)\Gamma(1-a_0)}{\Gamma(b_r-a_0)} \cdot \prod_{\substack{1 \leq j \leq m \\ j \neq r}} \frac{\Gamma(a_j-b_r+1)\Gamma(b_j-a_j)}{\Gamma(b_j-b_r+1)} \cdot f_r, \end{aligned}$$

where $1 \leq r \leq m$.

Proof. By pulling back $u\varphi_0$ under ι_0 , we can show the first claim. We prove the second one. On Δ_r , we have

$$u = e^{-\pi\sqrt{-1}(b_r - a_r)}(t_r - t_{r-1})^{b_r - a_r}(1 - xt_m)^{-a_0} \\ \cdot \prod_{j=1}^m t_j^{a_j - b_{j+1}} \cdot \prod_{\substack{1 \leq j \leq m \\ j \neq r}} (t_{j-1} - t_j)^{b_j - a_j},$$

where the argument of each factor is zero on $\iota_r(\sigma_r) \subset D_r$. We consider the pull back of $u\varphi_0$ under ι_r :

$$u(\iota_r(z)) = e^{-\pi\sqrt{-1}(b_r - a_r)} \cdot x^{-b_r} \cdot u_r(z), \\ \iota_r^* \varphi_0 = -x \cdot \phi_r.$$

By Proposition 4.2, we thus have

$$\int_{\Delta_r} u\varphi_0 = -e^{-\pi\sqrt{-1}(b_r - a_r)} x^{1-b_r} \int_{\tilde{\Delta}_r} u_r \phi_r = e^{-\pi\sqrt{-1}(b_r - a_r - 1)} \cdot (\Gamma\text{-factors}) \cdot f_r.$$

□

Remark 4.4. For $1 \leq r \leq m$, the twisted cycle Δ_r is different from the regularization of $D_r \otimes u$ as elements in $H_m(\mathcal{C}_\bullet(M, u))$.

Remark 4.5. Let $\iota'_r : M_r \rightarrow M_0$ ($1 \leq r \leq m$) be the map defined as

$$\iota'_r(z_1, \dots, z_m) := (w_1, \dots, w_m); \quad w_r = \frac{z_r}{x \prod_{j \neq r} z_j}, \quad w_s = z_s \quad (s \neq r).$$

Then it is easy to see that $\iota_r = \iota_0 \circ \iota'_r$.

The replacement $u \mapsto u^{-1} = 1/u$ and the construction same as Δ_k give the twisted cycle Δ_k^\vee which represents an element in $H_m(\mathcal{C}_\bullet(M, u^{-1}))$. We obtain the intersection numbers of the twisted cycles $\{\Delta_k\}_{k=0}^m$ and $\{\Delta_k^\vee\}_{k=0}^m$.

Theorem 4.6. (i) For $k \neq l$, we have $I_h(\Delta_k, \Delta_l^\vee) = 0$.

(ii) The self-intersection numbers of Δ_k 's are as follows:

$$I_h(\Delta_0, \Delta_0^\vee) = \prod_{i=1}^m \frac{\alpha_i(1 - \beta_i)}{(1 - \alpha_i)(\alpha_i - \beta_i)}, \\ I_h(\Delta_r, \Delta_r^\vee) = \prod_{\substack{1 \leq j \leq m \\ j \neq r}} \frac{\alpha_j(\beta_r - \beta_j)}{(\beta_r - \alpha_j)(\alpha_j - \beta_j)} \cdot \frac{\alpha_0 - \beta_r}{(1 - \beta_r)(\alpha_0 - 1)} \quad (1 \leq r \leq m),$$

$$\text{where } \alpha_j := e^{2\pi\sqrt{-1}a_j}, \quad \beta_j := e^{2\pi\sqrt{-1}b_j}.$$

Proof. This theorem can be also shown similarly to Theorem 4.6 of [5] or Theorem 4.6 of [6]. □

Corollary 4.7. Under the condition (3), the twisted cycles $\Delta_0, \dots, \Delta_m$ form a basis of $H_m(\mathcal{C}_\bullet(M, u))$

Proof. The determinant of the intersection matrix $H := (I_h(\Delta_i, \Delta_j))_{i,j=0,\dots,m}$ does not vanish. □

Remark 4.8. In Section 3 of [10], there are the twisted cycles $D_1^{(0)}, \dots, D_m^{(0)}, D_{m+1}^{(0)}$ which correspond to the solutions f_1, \dots, f_m, f_0 , respectively. By the variable change

$$p : (t_1, \dots, t_m) \mapsto \left(\frac{1}{t_1}, \dots, \frac{1}{t_m} \right),$$

our integral representation (2) coincides with that in [10]. It is easy to see that

$$\Delta_0 = (-1)^m p_*(D_{m+1}^{(0)}), \quad \Delta_r = (-1)^m \frac{\beta_r - \alpha_0}{\alpha_0(\beta_r - 1)} p_*(D_r^{(0)}) \quad (1 \leq r \leq m)$$

as elements in $H_m(\mathcal{C}_\bullet(M, u))$.

5. TWISTED PERIOD RELATIONS

The compatibility of the intersection forms and the pairings obtained by integrations (see [4]) implies twisted period relations:

$$C = \Pi_\omega {}^t H^{-1} {}^t \Pi_{-\omega},$$

where $\Pi_{\pm\omega}$ are defined as

$$\Pi_\omega := \left(\int_{\Delta_j} u \varphi_i \right)_{i,j}, \quad \Pi_{-\omega} := \left(\int_{\Delta_j^\vee} u^{-1} \varphi_i \right)_{i,j},$$

C and H are the intersection matrices (see the proof of Corollaries 3.2 and 4.7). Comparing the (i, j) -entries of both sides, we obtain the following theorem.

Theorem 5.1. *We have*

$$(9) \quad I_c(\varphi_i, \varphi_j) = \sum_{k=0}^m \frac{1}{I_h(\Delta_k, \Delta_k^\vee)} \cdot \int_{\Delta_k} u \varphi_i \cdot \int_{\Delta_k^\vee} u^{-1} \varphi_j.$$

By using our results, we can reduce the twisted period relations (9) to quadratic relations among ${}_{m+1}F_m$'s. We write down one of them as a corollary.

Corollary 5.2. *The equality (9) for $i = j = 0$ is reduced to*

$$\begin{aligned} \prod_{l=1}^m \frac{b_l - b_0}{a_l - b_0} &= \prod_{l=1}^m \frac{b_l}{a_l} \cdot {}_{m+1}F_m \left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}; x \right) \cdot {}_{m+1}F_m \left(\begin{matrix} -\mathbf{a} \\ -\mathbf{b} \end{matrix}; x \right) \\ &\quad + \sum_{r=1}^m x^2 \cdot \frac{a_0(a_0 - b_r)(b_r - a_r)}{b_r(b_r^2 - 1)} \cdot \prod_{\substack{1 \leq l \leq m \\ l \neq r}} \frac{a_l - b_r}{b_l - b_r} \\ &\quad \cdot {}_{m+1}F_m \left(\begin{matrix} \mathbf{a}^{r,+} \\ \mathbf{b}^{r,+} \end{matrix}; x \right) \cdot {}_{m+1}F_m \left(\begin{matrix} \mathbf{a}^{r,-} \\ \mathbf{b}^{r,-} \end{matrix}; x \right), \end{aligned}$$

where

$$\begin{aligned} \mathbf{a} &:= (a_1, \dots, a_{m+1}), \quad \mathbf{b} := (b_1, \dots, b_m), \\ \mathbf{a}^{r,\pm} &:= (1, \dots, 1) \pm (a_1 - b_r, \dots, a_{m+1} - b_r), \\ \mathbf{b}^{r,\pm} &:= (1, \dots, 1) \pm (b_1 - b_r, \dots, \pm 1 - b_r, \dots, b_m - b_r). \end{aligned}$$

Proof. By Theorem 4.3, we can express the integrals in (9) as products of Γ -factors and ${}_{m+1}F_m$. By Theorems 3.1, 4.6, and the formula $\Gamma(w)\Gamma(1-w) = \pi/\sin(\pi w)$, we obtain the corollary. \square

Remark 5.3. *If we assume the condition (3) and*

$$a_i - a_j \notin \mathbb{Z} \quad (0 \leq i < j \leq m),$$

then ψ_0, \dots, ψ_m also form a basis of $H^m(M, \nabla_\omega)$, because of Theorem 3.1. Considering $I_c(\varphi_i, \psi_j)$, $I_c(\psi_i, \varphi_j)$, or $I_c(\psi_i, \psi_j)$, we obtain other twisted period relations.

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KOBE UNIVERSITY, KOBE
657-8501, JAPAN

E-mail address: y-goto@math.kobe-u.ac.jp